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# On the fusion of face and vertex models

Yu-Kui Zhou†‡ and Bo-Yu Hou‡

† CCAST, PO Box 8730, Beijing, People's Republic of China

‡ Institute of Modern Physics, Xibei University, Xian 710069, People's Republic of China

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**Abstract.** Some new solutions of the Yang–Baxter or star-triangle relations are found using the fusion procedure. For Belavin's $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric model the relation between the fusion procedure of the vertex and face models is also constructed.

## 1. Introduction

Great progress has been made on the resolution of integrable models in statistical mechanics and quantum field theory [1-8]. The important mathematical structure that ensures the exact solvability of these models is the so-called Yang–Baxter relation (YBR) or the star-triangle relation (STR) [9, 10]. Very recently it also has been pointed out that the solutions of the YBR or STR may be used to construct the braid group, link invariants and operator algebras [11-14].

In the study of the YBR or STR the so-called fusion procedure [15] was developed to generate new integrable models (fusion models) corresponding to the group invariant solutions of the YBR or STR from a known model. In [15] the fusion procedure for the rational function solution of the YBR has been explicitly constructed. After this work several works devoted to the application of the fusion procedure to other exactly solvable models have been published. In particular, in [16] the fusion models have established from Baxter's eight-vertex model [1] and Belavin's $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric vertex model [17, 18]. The fusion procedure of the face models (SOS or IRF model) have also been considered in [19] for the ABF SOS model [20] and in [21] for the JMO IRF model [22]. These give symmetric tensors of the affine Lie algebra $A_{n-1}^{(1)}$ family of solvable lattice models [19, 20]. The vertex-IRF correspondence, representing the relation of the vertex model to its face model proposed first in [23] for the Baxter eight-vertex model and extended to the Belavin $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric vertex model in [22], has been established for the fusion of the Baxter eight-vertex model.

The present paper is directly motivated by these latest works. We give a unified consideration of the fusion procedure of the rational function solution (RFS) [15, 24], the triangle function solution (TFS) [25-27] and the elliptic function solution (EFS) [22] of the STR in face models. We construct some new fusion models corresponding to antisymmetric tensors and symmetric-antisymmetric tensors of the $A_{n-1}^{(1)}$ family of solvable lattice models and extend the vertex-IRF correspondence to the fusion models of the Belavin $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric model.

For the face models the role of the YBR is played by the STR [1, 28]. In this paper the solutions of the YBR are referred to as the Boltzmann weights of vertex models and those of the STR are referred to as the weights of face models. In § 2, we describe briefly the RFS [15, 24], TFS [25-27] and EFS [22] of the STR. They are elementary blocks for the fusion procedure. In § 3, we formulate uniformly the fusion procedure for these elementary blocks and obtain the fusion models. In § 4, we recapitulate the fusion procedure of the Belavin $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric vertex model [18]. Then we discuss the relation between the fusion procedures for the vertex and JMO IRF models. In the final section we give a brief discussion.

## 2. Elementary blocks

To describe the elementary blocks we prepare briefly some notation. For details the readers is referred to [29]. Let $\Lambda_\mu$ and $\hat{\mu}$ $(0 \leqslant \mu \leqslant n-1)$ be respectively the fundamental weights and elementary vectors of affine Lie algebra $A_{n-1}^{(1)}$. Using a set of distinguished basis $\varepsilon_\mu$ $(0 \leqslant \mu \leqslant n-l)$ with $\langle \varepsilon_\mu, \varepsilon_\nu \rangle = \delta_{\mu\nu}$, the elementary vectors are $\hat{\mu} = \Lambda_{\mu+1} - \Lambda_\mu = \varepsilon_\mu - (1/n)(\varepsilon_0 + \varepsilon_1 + \ldots + \varepsilon_{n-1})$. We define $a^{\mu\nu} = \langle a+\rho, \varepsilon_\mu - \varepsilon_\nu \rangle = a^\mu - a^\nu$, where $\rho = \Lambda_0 + \Lambda_1 + \ldots + \Lambda_{n-1}$ and $a$ is an element of the generic complex weights $\Sigma\, \mathbb{C}\Lambda_\mu + \Sigma\, \mathbb{Z}\hat{\mu}$.

For each state configuration $(a+\hat{\mu}, a+\hat{\mu}+\hat{\nu}, a+\hat{\nu}', a)$ of four sites surrounding a face we define a Boltzmann weight of an IRF model. We write these Boltzmann weights as the elements of a $W$ matrix [30]:

$$W(a|z)_{\nu\mu}^{\nu'\mu'} = W\left(\begin{matrix} a & a+\hat{\nu}' \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{matrix}\middle| z\right) \tag{2.1}$$

with

$$\hat{\nu}' + \hat{\mu}' = \hat{\nu} + \hat{\mu}. \tag{2.2}$$

It satisfies the following STR [22, 30]:

$$\begin{aligned} W(a+\hat{\mu}_3|z_1-z_2)_{\mu_1\mu_2}^{\lambda_1\lambda_2} & W(a|z_1)_{\lambda_1\mu_3}^{\nu_1\lambda_3} W(a+\hat{\nu}_1|z_2)_{\lambda_2\lambda_3}^{\nu_2\nu_3} \\ &= W(a|z_2)_{\mu_2\mu_3}^{\lambda_2\lambda_3} W(a+\hat{\lambda}_2|z_1)_{\mu_1\lambda_3}^{\lambda_1\nu_3} W(a|z_1-z_2)_{\lambda_1\lambda_2}^{\nu_1\nu_2} \end{aligned} \tag{2.3}$$

where the double $\lambda$ indices imply summations over 0 to $n-1$.

The solutions of the STR (2.3) have three functional forms. They are [15, 22, 24-27, 31]

$$W\left(\begin{matrix} a & a+\hat{\nu}' \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{matrix}\middle| z\right)$$

$$= \begin{cases} z\delta_{\nu\nu'} + \delta_{\mu\nu'} & \text{RFS} \quad (2.4a) \\ \dfrac{\sin(w(z+1))}{\sin w}\delta_{\mu\nu} + k_{\mu\nu}(z)(1-\delta_{\mu\nu}) & \text{TFS} \quad (2.4b) \\ \dfrac{h[(a+\hat{\mu})^{\mu\nu}-1-z]}{h[(a+\hat{\mu})^{\mu\nu}-1]}\delta_{\mu\nu'} + \dfrac{h(z)}{h(1)}\dfrac{h[(a+\hat{\mu})^{\mu\nu}]}{h[(a+\hat{\mu})^{\mu\nu}-1]}\delta_{\nu\nu'} & \text{EFS} \quad (2.4c) \end{cases}$$

where

$$k_{\mu\nu}(z) = G_{\mu\nu}(z)\delta_{\mu\nu'} + \frac{\sin zw}{\sin w} G_{\nu\mu}(1)\delta_{\nu\nu'}$$

$$G_{\mu\nu}(z) = \exp[izw\,\mathrm{sgn}(\mu-\nu)] \tag{2.5a}$$

or

$$G_{\mu\nu}(z)=\exp\left[\mathrm{i}\,zw\left(\frac{\nu-\mu}{n}+\mathrm{sgn}(\mu-\nu)\right)\right] \tag{2.5b}$$

$$h(z)=\prod_{i=0}^{n-l}\theta^{(i)}(z)\left(\prod_{i=l}^{n-l}\theta^{(i)}0)\right)^{-1} \tag{2.5c}$$

$$\theta^{(i)}(z)=\sum_{m\in\mathbb{Z}}\exp[\mathrm{i}\pi(m+\tfrac{1}{2}-i/n)^2\tau+\mathrm{i}2\pi(m+\tfrac{1}{2}-i/n)(zw+\tfrac{1}{2})]. \tag{2.5d}$$

The Boltzmann weights (1.4$a$, $b$) are independent of the generic complex weight $a$ of the $A^{(1)}_{n-1}$. In fact the $W$ matrices for the RFS (1.4$a$) and TFS (1.4$b$) are just equal to their vertex Boltzmann weight matrices. Therefore, (2.1) could be considered as a direct correspondence of the face model to the vertex model for the RFS(1.4$a$) or TFS(1.4$b$). The EFS(1.4$c$) correspondence to its vertex model or the Belavin $\mathbb{Z}_n\times\mathbb{Z}_n$ symmetric vertex model [17, 18] by the vertex-IRF correspondence [22] instead of such a direct correspondence. The $W$ matrix of the EFS(1.4$c$), however, always represents the face Boltzmann weight matrix.

## 3. Fusion procedure

In this section we study the fusion procedure for these face models. First, we introduce

$$W^{\mu\mu'}_{M,1}(a|z)^{\nu'_1\ldots\nu'_M}_{\nu_1\ldots\nu_M}=W(a|z)^{\nu'_1\lambda_2}_{\nu_1\mu}W(a+\hat{\nu}'_l|z+1)^{\nu'_2\lambda_3}_{\nu_2\lambda_2}\ldots W\left(a+\sum_{s=1}^{M-1}\hat{\nu}'_s\,\middle|\,z+M-l\right)^{\nu'_M\mu'}_{\nu_M\lambda_M} \tag{3.1a}$$

$$W_{M,N}(a|z)^{\nu'_1\ldots\nu'_M;\mu'_N\ldots\mu'_1}_{\nu_1\ldots\nu_M;\mu_N\ldots\mu_1}=\left(W^{\mu_N\mu'_N}_{M,1}\left(a+\sum_{s=1}^{N-1}\hat{\mu}_s\,\middle|\,z-N+1\right)\ldots W^{\mu_2\mu'_2}_{M,1}(a+\hat{\mu}_1|z-1)\,W^{\mu_1\mu'_1}_{M,1}(a|z)\right)^{\nu'_1\ldots\nu'_M}_{\nu_1\ldots\nu_M}. \tag{3.1b}$$

For two integers $M$, $N\geqslant 2$ we define

$$W^{\alpha\beta}_{M,N}\begin{pmatrix} a & a+\nu' \\ a+\mu & a+\mu+\nu \end{pmatrix}\!\Big|\,z\Big)=Y^{\alpha}_{M}Y^{\beta}_{N}W_{M,N}(a|z)^{\nu'_1\ldots\nu'_M;\mu'_N\ldots\mu'_1}_{\nu_1\ldots\nu_M;\mu_N\ldots\mu_1}Y^{\alpha}_{M}Y^{\beta}_{N} \tag{3.2}$$

where $\alpha$, $\beta=+,-$ and also

$$\mu=\sum_{s=1}^{N}\hat{\mu}_s \qquad \nu=\sum_{s=1}^{M}\hat{\nu}_s \qquad \nu'=\sum_{s=1}^{M}\hat{\nu}'_s.$$

These operators $Y^{\alpha}(\alpha=+,-)$ act on $W_{M,N}(a|z)$ from the left or the right as follows. For example:

$$Y^{\alpha}_{N}W_{M,N}(a|z)^{\nu'_1\ldots\nu'_M;\mu'_N\ldots\mu'_1}_{\nu_1\ldots\nu_M;\mu_N\ldots\mu_1}=\frac{1}{N!}\sum_{P}\varepsilon^{\alpha}_{P}W_{M,N}(a|z)^{\nu'_1\ldots\nu'_M;\mu_N\ldots\mu_1}_{\nu_1\ldots\nu_M;\mu_{P(N)}\ldots\mu_{P(1)}}. \tag{3.3}$$

Here $P$ is the permutation on $N$ objects that sends the ordered set $(\hat{\mu}_N\ldots\hat{\mu}_2\hat{\mu}_1)$ with $\mu_1<\mu_2<\ldots<\mu_N$ $(2\leqslant N\leqslant n)$ and $\mu_l\leqslant\mu_2\leqslant\ldots\leqslant\mu_N$ $(N\geqslant 2)$, respectively, for $\alpha=-$ and $\alpha=+$, to the ordered set $(\mu_{P(N)}\ldots\mu_{P(2)}\mu_{P(l)})$. $\varepsilon^{-}_{P}=+1$ $(-1)$ for $P$ being even (odd) and $\varepsilon^{+}_{P}=+1$. If $p$ is the permutation on $(\mu'_N\ldots\mu'_2\mu'_1)$ or $(\nu'_1\nu'_2\ldots\nu'_M)$, (3.3) defines the $Y^{\alpha}_{N}$ or $Y^{\alpha}_{M}$ acting on $W_{M,N}(a|z)$ from the right.

Equation (3.2) defines the fusion Boltzmann weights. In the following we show that they satisfy the STR.

From (2.4) we have

$$W(a|-1) = B^- Y_2^- \tag{3.4a}$$

$$W(a|1) = Y_2^+ B^+ \tag{3.4b}$$

$$Y_2^\alpha = \tfrac{1}{2} \sum_{\nu,\mu} (E_{\nu\nu} \otimes E_{\mu\mu} + \alpha E_{\nu\mu} \otimes E_{\mu\nu}) \tag{3.4c}$$

where $B^+$ and $B^-$ are invertible matrices for almost all $w$. They have the following forms:

$$B^- = -\sum_{\nu} E_{\nu\nu} \otimes E_{\nu\nu} + \sum_{\mu,\nu} W(a|-1)^{\nu\mu}_{\nu\mu} E_{\nu\mu} \otimes E_{\mu\mu} \tag{3.5a}$$

$$B^+ = \sum W(a|1)^{\nu\mu}_{\nu\mu} E_{\nu\nu} \otimes E_{\mu\mu} \tag{3.5b}$$

where $E_{\mu\nu}$ is a $n \times n$ matrix with $(E_{\nu\mu})_{\nu'\mu'} = \delta_{\nu\nu'} \delta_{\mu\mu'}$. The $W(a|\pm 1)^{\nu\mu}_{\nu\mu}$ can be obtained from (2.1) and (2.4).

With the help of (3.4) and the STR (2.3) we can obtain the following proposition.

*Proposition 1.*

$$Y_M^\alpha Y_N^\beta W_{M,N}(a|z)^{\nu_1' \cdots \nu_M'; \mu_N' \cdots \mu_1'}_{\nu_1 \cdots \nu_M; \mu_N \cdots \mu_1} Y_M^\alpha Y_N^\beta$$

$$= \begin{cases} W_{M,N}(a|z)^{\nu_1' \cdots \nu_M'; \mu_N' \cdots \mu_1'}_{\nu_1 \cdots \nu_M; \mu_N \cdots \mu_1} Y_M^+ Y_N^+ & \text{for } \alpha = \beta = + \\ Y_M^- Y_N^- W_{M,N}(a|z)^{\nu_1' \cdots \nu_M'; \mu_N' \cdots \mu_1'}_{\nu_1 \cdots \nu_M; \mu_N \cdots \mu_1} & \text{for } \alpha = \beta = - \\ Y_M^- W_{M,N}(a|z)^{\nu_1' \cdots \nu_M'; \mu_N' \cdots \mu_1'}_{\nu_1 \cdots \nu_M; \mu_N \cdots \mu_1} Y_N^+ & \text{for } \alpha = -\beta = - \\ Y_N^- W_{M,N}(a|z)^{\nu_1' \cdots \nu_M'; \mu_N' \cdots \mu_1'}_{\nu_1 \cdots \nu_M; \mu_N \cdots \mu_1} Y_M^+ & \text{for } \alpha = -\beta = +. \end{cases}$$

By a repeated use of (2.3) the STR of the $W_{M,N}(a\, z)$ follows. Multiplying this STR from the left and the right by $Y_M^\alpha$ and $Y_N^\beta$, and also with the help of proposition 1, we can obtain the following theorem.

*Theorem 1.* The Boltzmann weight $W_{M,N}\left(\begin{smallmatrix} a & a+\nu' \\ a+\mu & a+\mu+\nu \end{smallmatrix}\middle| z\right)$ satisfies the STR (three integers $M$, $N$, $L \geq 2$):

$$\sum_\lambda W^{\alpha\sigma}_{M,N}\begin{pmatrix} a+\mu_3 & a+\mu_3+\lambda \\ a+\mu_2+\mu_3 & a+\mu_1+\mu_2+\mu_3 \end{pmatrix}\!\left| z_1 - z_2\right) W^{\alpha\beta}_{M,L}\begin{pmatrix} a & a+\nu_1 \\ a+\mu_3 & a+\mu_3+\lambda \end{pmatrix}\!\left| z_1\right)$$

$$\times W^{\sigma\beta}_{N,L}\begin{pmatrix} a+\nu_1 & a+\nu_1+\nu_2 \\ a+\mu_3+\lambda & a+\mu_1+\mu_2+\mu_3 \end{pmatrix}\!\left| z_2\right)$$

$$= \sum_\lambda W^{\sigma\beta}_{N,L}\begin{pmatrix} a & a+\lambda \\ a+\mu_3 & a+\mu_2+\mu_3 \end{pmatrix}\!\left| z_2\right)$$

$$\times W^{\alpha\beta}_{M,L}\begin{pmatrix} a+\lambda & a+\nu_1+\nu_2 \\ a+\mu_2+\mu_3 & a+\mu_1+\mu_2+\mu_3 \end{pmatrix}\!\left| z_1\right)$$

$$\times W^{\alpha\sigma}_{M,N}\begin{pmatrix} a & a+\nu_1 \\ a+\lambda & a+\nu_1+\nu_2 \end{pmatrix}\!\left| z_1 - z_2\right). \tag{3.6}$$

In definition (3.2) these superscript and subscript $\mu$ and $\nu$ are elementary vectors of $A^{(1)}_{n-1}$. If a summation $\mu$ over $N$ elementary vectors is given, we can find the only set $(\hat{\mu}_1\hat{\mu}_2\ldots\hat{\mu}_N)$ satisfying $\mu=\hat{\mu}_1+\hat{\mu}_2+\ldots+\hat{\mu}_N$. So the summation over $P$ in (3.3) is equivalent to the summation over all $\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_N$ keeping $\mu=\hat{\mu}_1+\hat{\mu}_2+\ldots+\hat{\mu}_N$. This shows that the weight $W_{M,N}$ in (3.2) only depends on $\mu$, $\nu$, $\mu'$ and $\nu'$ instead of these $\hat{\mu}_i$, $\hat{\nu}_i$, $\hat{\mu}'_i$ and $\hat{\nu}'_i$. As $W(a|z)^{\mu'\nu'}_{\mu\nu}$ has $\hat{\mu}'+\hat{\nu}'=\hat{\mu}+\hat{\nu}$, $W_{M,N}$ has $\mu+\nu=\mu'+\nu'$. We have omitted the $\mu'$ in $W^{\alpha\beta}_{M,N}$.

So far we have given the fusion procedure for these IRF models (2.4). The fusion Boltzmann weights are given by (3.2). They are symmetric tensors, antisymmetric tensors and symmetric-antisymmetric tensors, for $(\alpha,\beta)=(+,+)$, $(\alpha,\beta)=(-,-)$ and $(\alpha,\beta)=(+,-)$, respectively, of the $A^{(1)}_{n-1}$ family of solvable lattice models. The symmetric tensors for the EFS (2.4$c$) have been constructed in [21]. The RFS (1.4$a$) as a vertex model has been studied and developed the fusion procedure in [15].

The fusion procedure of a TFS [27], which is different from (2.4$b$) by replacing $G_{\nu\mu}(1)$ with 1 in the factor $k_{\mu\nu}(z)$, has been considered in [32]. The $Y^{\alpha}$ in (3.3) is, however, different from the projector used in [32].

## 4. Vertex–IRF correspondence

After the work in [15] the fusion procedure of the Belavin $\mathbb{Z}_n\times\mathbb{Z}_n$ symmetric vertex model was studied in [16]. In this section we construct the relation between the fusion model given there and our fusion model for the EFS.

We describe first the fusion procedure of the Belavin $\mathbb{Z}_n\times\mathbb{Z}_n$ symmetric vertex model [16]. Let

$$S(z)=\sum S(z)^{kl}_{ij}E_{ik}\otimes E_{jl} \tag{4.1$a$}$$

with [22, 33]

$$S(z)^{kl}_{ij}=\begin{cases}h(z)\theta^{(k-l)}(z+1)/\theta^{(k-i)}(l)\theta^{(i-l)}(z) & \text{as } i+j=k+l \bmod n\\ 0 & \text{otherwise}\end{cases} \tag{4.1$b$}$$

the matrix of vertex weights of the Belavin $\mathbb{Z}_n\times\mathbb{Z}_n$ symmetric model. Where the summation is over $i$, $j$, $k$ and $l$ from 0 to $n-1$. Let $M$ and $N$ be two positive integers and $V_i=\bar{V}_j=C^n$ $(1\leq i\leq N, 1\leq j\leq M)$, $V^{\otimes N}=V_1\otimes\ldots\otimes V_N$ and $\bar{V}^{\otimes M}=\bar{V}_1\otimes\ldots\otimes\bar{V}_M$. $S^{ij}(z)$ $(S^{i\bar{j}}(z))$ is the operator $S(z)$ acting on $V_i\otimes V_j$ $(V_i\otimes\bar{V}_j)$. The YBR is

$$S^{12}(z_1-z_2)S^{13}(z_1)S^{23}(z_2)=S^{23}(z_2)S^{13}(z_1)S^{12}(z_1-z_2). \tag{4.2}$$

From (4.1) we have [16]

$$\begin{aligned}&S(-1)=A^-P_2^- \qquad S(1)=P_2^+A^+\\ &P_2^\alpha=\tfrac{1}{2}\sum_{ij}(E_{ii}\otimes E_{jj}+\alpha E_{ij}\otimes E_{ji})\end{aligned} \tag{4.3}$$

where

$$A^-=-\sum_i E_{ii}\otimes E_{ii}+S(-1)$$

$$A^+=S(1).$$

We can show easily that $\det(A^{\mp})\neq 0$ as $\tau\to i\infty$ or $w\to 0$. This leads to $\det(A^{\mp})\neq 0$ for almost all $w$. Hence matrices $A^+$ and $A^-$ are invertible for almost all $w$.

We define the operator acting on $V^{\otimes N}\otimes \bar{V}_j$ by

$$S_{N\bar{j}}(z)=S^{1\bar{j}}(z-N+1)S^{2\bar{j}}(z-N+2)\ldots S^{N\bar{j}}(z) \tag{4.4a}$$

and the operator acting on $V^{\otimes N}\otimes V^{\otimes M}$ by

$$S_{NM}(z)=S_{N\bar{M}}(z)\ldots S_{N\bar{2}}(z+M-2)S_{N\bar{1}}(z+M-1). \tag{4.4b}$$

With the help of (4.2) and (4.3) we have the next proposition.

*Proposition 2*

$$S_{NM}(z)P^+_{1\ldots N}P^+_{1\ldots M}=P^+_{1\ldots N}P^+_{1\ldots M}S_{NM}(z)P^+_{1\ldots N}P^+_{1\ldots M}$$

$$P^-_{1\ldots N}P^-_{1\ldots M}S_{NM}(z)=P^-_{1\ldots N}P^-_{1\ldots M}S_{NM}(z)P^-_{1\ldots N}P^-_{1\ldots M}$$

$$P^-_{1\ldots N}S_{NM}(z)P^+_{1\ldots M}=P^-_{1\ldots N}P^+_{1\ldots M}S_{NM}(z)P^-_{1\ldots N}P^+_{1\ldots M}$$

$$P^-_{1\ldots M}S_{NM}(z)P^+_{1\ldots N}=P^-_{1\ldots M}P^+_{1\ldots N}S_{NM}(z)P^-_{1\ldots M}P^+_{1\ldots N}$$

where $P^\alpha_{1\ldots N}$ $(P^\alpha_{1\ldots M})$ denotes the projector on the space of symmetric tensor and antisymmetric tensor in $V^{\otimes N}(V^{\otimes M})$, respectively, for $\alpha=+$ and $\alpha=-$.

We define an operator acting on $V^{\otimes N}\otimes V^{\otimes M}$ by

$$S^{\alpha\beta}_{NM}(z)=P^\alpha_{1\ldots N}P^\beta_{1\ldots M}S_{NM}(z)P^\alpha_{1\ldots N}P^\beta_{1\ldots M}. \tag{4.5}$$

Using (4.2) and proposition 2 we have the following theorem.

*Theorem 2.* Fix a triple of integers $M, N, L\geqslant 2$. $S_{NM}(z)$ satisfies the YBR $(\alpha,\beta,\sigma=+,-)$

$$S^{\sigma\alpha}_{NM}(z_1-z_2)S^{\sigma\beta}_{NL}(z_1)S^{\alpha\beta}_{ML}(z_2)=S^{\alpha\beta}_{ML}(z_2)S^{\sigma\beta}_{NL}(z_1)S^{\sigma\alpha}_{NM}(z_1-z_2). \tag{4.6}$$

The matrix elements of the $S^{\alpha\beta}_{NM}(z)$ are the vertex weights of the fusion model. The $S^{\alpha\beta}_{NM}(z)$ $(\alpha=+$ or $-)$ have been obtained in [16].

The fusion vertex model $S^{\alpha\beta}_{NM}$ relates to the $W^{\beta\alpha}_{MN}$ in (3.2) by a vertex-IRF correspondence. In the following we construct this correspondence. This has been studied for $n=2$ in [19].

We introduce the interwining vectors (the superscipt t means the matrix transpose)

$$\phi^\mu_a(z)={}^{t}(\theta^{(0)}(z-na^\mu),\theta^{(1)}(z-na^\mu),\ldots,\theta^{(n-1)}(z-na^\mu)) \tag{4.7a}$$

$$\phi_{a,b}(z)^{\mu_1\cdots\mu_N}=\phi^{\mu_1}_a(z)\otimes\phi^{\mu_2}_{a+\hat{\mu}_1}(z-1)\otimes\ldots\otimes\phi^{\mu_N}_{a+\hat{\mu}_1+\ldots+\hat{\mu}_{N-1}}(z-N+1) \tag{4.7b}$$

and

$$\phi^\alpha_{a,b}(z)_N=P^\alpha_{1\ldots N}\phi_{a,b}(z)^{\mu_1\cdots\mu_N}Y^\alpha_N \tag{4.7c}$$

where the operator $Y^\alpha_N$ is defined in (3.3).

Using (4.7) and $\hat{\mu}^\nu=\langle\hat{\mu},\varepsilon_\nu\rangle$ we have the proposition 3.

*Proposition 3*

$$P^+_{1\ldots N}\phi_{a,b}(z)^{\mu_1\cdots\mu_N}Y^+_N=P^+_{1\ldots N}\phi_{a,b}(z)^{\mu_1\cdots\mu_N}$$

$$P^-_{1\ldots N}\phi_{a,b}(z)^{\mu_1\cdots\mu_N}Y^-_N=\phi_{a,b}(z)^{\mu_1\cdots\mu_N}Y^-_N$$

where $b=a+\hat{\mu}_1+\hat{\mu}_2+\ldots+\hat{\mu}_N$.

The so-called vertex-IRF correspondence for the EFS was first introduced by Baxter in the study of the eight-vertex model [23] and extended to the Belavin $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric vertex model by Jimbo *et al.* It is [22]

$$
{}^{t}S(z_1 - z_2)\phi_a^{\mu}(z_1)\otimes\phi_{a+\hat{\mu}}^{\nu}(z_2) = \sum_{\lambda} \phi_{a+\hat{\lambda}}^{\mu+\nu-\lambda}(z_1)\otimes\phi_a^{\lambda}(z_2)\, W\begin{pmatrix} a & a+\hat{\lambda} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{pmatrix} \Bigg| z_1 - z_2 \Bigg) \qquad (4.8)
$$

where $S(z_1 - z_2)$ and $W(\vdots\,|\,z_1 - z_2)$ are respectively given in (4.1) and (2.4$c$).

By a similar discussion to theorems 1 and 2, the correspondence (4.8) can be generalised to

$$
{}^{t}S_{NM}^{\alpha\beta}(z_1 - z_2)\phi_{a,b}^{\alpha}(z_1)_N \otimes \phi_{b,c}^{\beta}(z_2)_M = \sum_{d} \phi_{d,c}^{\alpha}(z_1)_N \otimes \phi_{a,d}^{\beta}(z_2)_M W_{M,N}^{\beta\alpha}\begin{pmatrix} a & d \\ b & c \end{pmatrix} \Bigg| z_1 - z_2 \Bigg). \qquad (4.9)
$$

This relation coincides with that in [19] for the case $\alpha = \beta = +$ and $n = 2$.

## 5. Brief discussion and conclusion

The present work gives a unified study for the fusion procedure of the RFS, TFS and EFS in (2.4), and also constructs the vertex-IRF correspondence of the fusion model for the Belavin $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetric model. For all these construction propositions 1-3 are a fundamental requirement. They represent the operators $Y^{\alpha}$ and $P^{\alpha}$ can pass through the fusion objects (Boltzmann weights or interwining vectors). This property is also used to establish a composite string [34] in a braid group and a parallel version of link invariants [35].

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